$\mathcal{PT}$ symmetry breaking in the presence of random, periodic, long-range hopping

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ABSTRACT

Over the past five years, open systems with balanced gain and loss have been investigated for extraordinary properties that are not shared by their closed counterparts. Non-Hermitian, Parity-Time ($\mathcal{PT}$) symmetric Hamiltonians faithfully model such systems. Such a Hamiltonian typically consists of a reflection-symmetric, Hermitian, nearest-neighbor hopping profile and a $\mathcal{PT}$-symmetric, non-Hermitian, gain and loss potential, and has a robust $\mathcal{PT}$-symmetric phase. Here we investigate the robustness of this phase in the presence of long-range hopping disorder that is not $\mathcal{PT}$-symmetric, but is periodic. We find that the $\mathcal{PT}$-symmetric phase remains robust in the presence of such disorder, and characterize the configurations where that happens. Our results are found using a tight-binding model, and we validate our predictions through the beam-propagation method.

Keywords: PT symmetry, disorder, waveguide arrays

1. INTRODUCTION

The requirement that the energy spectrum of a quantum mechanical Hamiltonian be real is usually satisfied by imposing the sufficient condition that it be Hermitian. However, this requirement may be relaxed in favor of other symmetry-driven constraints. These considerations have led to the study of a broad class of Hamiltonians which are invariant under combined parity and time-reversal ($\mathcal{PT}$) operations. These Hamiltonians, while not Hermitian, can have purely real spectra for a continuous, but finite range of their parameters. Although the spectrum of such a Hamiltonian may be purely real, due to its non-orthogonal eigenfunctions, the time-evolution of the corresponding quantum system is not unitary. However, when the spectrum is real, the violation of unitarity is bounded and periodic in time.

Over the past 18 years, since its inception by Bender et al.,[1] $\mathcal{PT}$-symmetric Hamiltonians have been a topic of great theoretical interest. Although their original intent was to define a new fundamental quantum theory,[2] in recent years, the study of $\mathcal{PT}$ symmetric systems has gained much interest for its applications to open systems with balanced gain and loss.[3] Typically, in such a system, the parity symmetry denotes a reflection symmetry in its spatial arrangement, and when balanced gain and loss (which lead to non-Hermiticity) are introduced, the resultant open system is intrinsically $\mathcal{PT}$-symmetric. For small gain and loss rates, the eigenvalues of the $\mathcal{PT}$-symmetric Hamiltonian describing such a system remain real; however, when the strength of the gain (or loss) exceeds a value known as the $\mathcal{PT}$-symmetry breaking threshold, two or more of its eigenvalues become degenerate and then complex-conjugate pairs. This emergence of complex conjugate eigenvalues is called $\mathcal{PT}$ symmetry breaking and leads to unbounded, exponential violation of unitarity in the time-development of the system. Thus, below the threshold, the system remains in a quasi equilibrium state, and above it, the system is far removed from equilibrium. It is the existence of a finite threshold which is the hallmark of a $\mathcal{PT}$ symmetry breaking transition and the accompanying phenomena that occur at the exceptional point.[4]

While $\mathcal{PT}$-symmetric Hamiltonians may, in general, be continuous in their degrees of freedom, it is the discrete $\mathcal{PT}$-symmetric systems that have proven to be the experimentally implementable ones. Recent developments in the fabrication techniques of optical devices have led to the ability to easily create and control arrays of coupled optical waveguides, and the couplings of these arrays can be tuned to match the dynamics of a large variety of different tight-binding Hamiltonians.[5–7] Controlled loss and gain can also be implemented

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relatively straightforwardly, allowing the observable dynamics to extend into the non-Hermitian realm,[8–10] while cost-effective fabrication processes promise to further boost the importance of such tunable systems.[11] The interactions within these systems, in one dimension, are dominated by nearest-neighbor hopping; however, for certain one or two dimensional waveguide geometries, higher order (or long-range) hopping can be made relevant. These systems have been greatly successful in demonstrating a variety of quantum phenomena, including localization[7,12,13] and the presence of edge modes[14,15] in a classical, and experimentally viable setting.

One dimensional models which exhibit the edge states and topological properties, such as the Su-Schrieffer-Heeger[16] (SSH) model or the André-Aubry-Harper[17,18] (AAH) model, typically have a periodic modulation of on-site potential or nearest-neighbor hopping. This modulation ruins the reflection symmetry of such finite lattice models, but such models, nonetheless, show a positive $\mathcal{PT}$-symmetry breaking threshold under the right circumstances.[19,20] These results raise the possibility that a periodic, random, long-range hopping disorder will, despite lacking reflection symmetry, lead to a positive $\mathcal{PT}$-symmetry breaking threshold.

What models of periodic, non-$\mathcal{PT}$-symmetric disorder retain a nonzero $\mathcal{PT}$ threshold? We address this question in the following section. In Sec. 2.1 we first recall the results for $\mathcal{PT}$-symmetry breaking threshold in an N-site uniform lattice with a pair of gain-loss potentials at reflection-symmetric sites. In Sec. 2.2 we show that the presence of an on-site disorder or nearest-neighbor hopping disorder with appropriate period p leads to a finite threshold for suitable gain-loss locations. In Sec. 2.3 we show that these results, obtained by using a tight-binding lattice approximation, are valid for electromagnetic wave propagation under paraxial approximation in a coupled waveguide array with appropriate gain and loss index profiles. Our results are generalized to long-range hopping disorder – next-nearest-neighbor hopping and even higher-order hopping – in Sec. 2.4. Section 3 has a brief conclusion.

2. LATTICE MODELS WITH PERIODIC, HOPPING DISORDER

In this section, we first establish the notation and recall results for the $\mathcal{PT}$-symmetric phase in a uniform, tight-binding lattice with open boundary conditions.

2.1 The Uniform Lattice

The starting point for our model is an array of N waveguides with constant, nearest-neighbor hopping $J > 0$ that is determined by the overlap of exponentially decaying electric fields in the region between two adjacent waveguides. Thus, J defines the natural energy or frequency scale for the lattice. We may express this system as a tight-binding lattice with a basis which is defined by the waveguide labels $m = 1, 2, ..., N$. The uniform-hopping Hamiltonian is given by

$$H_U = -J \sum_{m=1}^{N-1} |m\rangle \langle m+1| + \text{h.c.} = H_U^\dagger \tag{1}$$

where $|m\rangle$ represents a state fully localized in waveguide m. Here, open boundary conditions are denoted by the fact that there are no hopping elements between sites 1 and N. The parity operation on this lattice is given by $\mathcal{P} : m \to \bar{m} = N + 1 - m$ and represents reflection about the lattice center, whereas time-reversal operation corresponds to complex conjugation, $\mathcal{T} = *$. Thus far, the system described is Hermitian; however, suppose we introduce a controlled gain with strength γ into the waveguide m_0, and maintain a controlled loss of equal strength γ at the spatially symmetric waveguide $\bar{m}_0 = N - m_0 + 1$. In the tight-binding Hamiltonian, this corresponds to a pair of complex-conjugate imaginary potentials located at the aforementioned locations, and is given by

$$\Gamma = \mathrm{i}\gamma(|m_0\rangle \langle m_0| - |\bar{m}_0\rangle \langle \bar{m}_0|) = -\Gamma^\dagger \,. \tag{2}$$

The non-Hermitian Hamiltonian $H(\gamma) = H_U + \Gamma$ is, nonetheless, $\mathcal{PT}$-symmetric, $\mathcal{PT}H\mathcal{PT} = H$. Since the hopping Hamiltonian also commutes with the parity operator alone, its eigenfunctions $\psi_\alpha(k)$ are interlaced even and odd functions, i.e. $\mathcal{P}\psi_\alpha = (-1)^{\alpha-1}\psi_\alpha$. Starting from zero, as the gain-loss strength γ is increased, the eigenvalues of $H(\gamma)$ remain real and its eigenfunctions are simultaneous eigenfunctions of $\mathcal{PT}$, until the $\mathcal{PT}$ symmetry is broken at $\gamma = \gamma_{PT}$, resulting in complex conjugate eigenvalues for $H(\gamma)$. Figure 1 summarizes the results for the $\mathcal{PT}$-symmetry breaking threshold $\gamma_{PT}(m_0)$ for even and odd lattice sizes.[21] It shows that the $\mathcal{PT}$

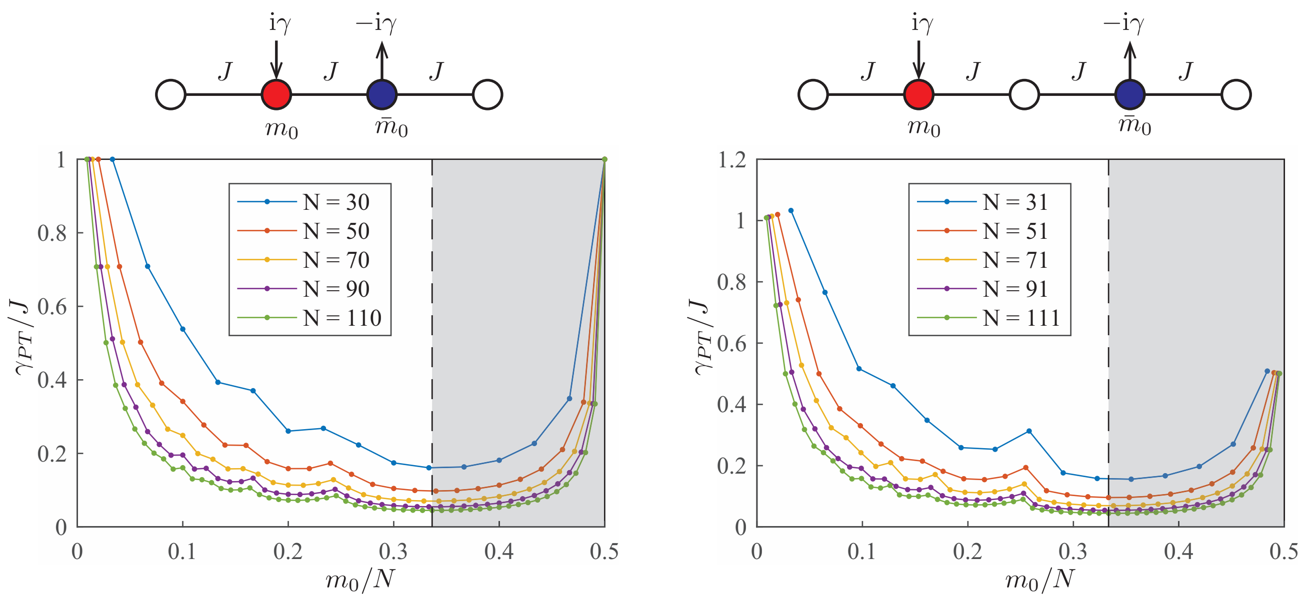

Figure 1. (Top) Schematic of an even ($N = 4$) and odd ($N = 5$) lattice with open boundary conditions, a uniform hopping $J > 0$, and a pair $\pm i\gamma$ of gain-loss potentials at mirror symmetric locations $m_0 \leq N/2$ and $\bar{m}_0 = N + 1 - m_0$ respectively. (Bottom) $\mathcal{PT}$-breaking threshold γ_{PT}/J for even (left) and odd (right) cases as a function of gain location $m_0/N < 0.5$ for various system sizes N. The shaded region indicates the fraction of the lattice for which the $\mathcal{PT}$-threshold decays monotonically with the distance between the gain and loss potentials, and thus the results are what are expected in an infinite lattice. In the non-shaded area, $m_0/N \leq 0.35$, we see that due to the boundary effects, the $\mathcal{PT}$-threshold increases even as the distance between gain and loss sites increases. The threshold reaches a maximum value, $\gamma_{PT}/J = 1$, for $N \gg 1$ when the gain and loss sites are farthest removed from each other.

breaking threshold is of order unity when the gain and loss are closest to each other, $m_0/N \sim 1/2$. When, the gain/loss separation distance $|\bar{m}_0 - m_0|$ is increased from 0, as might be expected, the $\mathcal{PT}$-breaking threshold decreases rapidly. For an infinite lattice, this would be the only such trend; however, at a certain fractional distance from the edges, the boundary effects begin to manifest themselves, and the $\mathcal{PT}$-breaking threshold reaches a maximum $\gamma_{PT}/J \sim 1$ when the gain and the loss are farthest from each other. Open boundary conditions are instrumental to this unexpected strengthening of the $\mathcal{PT}$-symmetry breaking threshold.

2.2 Introduction of Periodic Disorder

Motivated by Hermitian SSH and AAH models and periodic-disorder-related phenomena, we seek to alter the on-site potentials and nearest-neighbor or higher-order hopping elements of this system in a periodic fashion. The Hermitian Hamiltonians corresponding to these perturbations are given by

$$H_k = -\sum_{m=1}^{N-1} C_k(m) \left|m\right\rangle \left\langle m+k\right| + \text{h.c.} \tag{3}$$

where $k \geq 0$, $C_k(m)$ denotes the hopping amplitude between sites m and $m+k$. When $k = 0$, the perturbation represents periodic variation of the on-site potential. For a given value k, we specify a sequence with length p_k of random numbers $\{r_{k,1}, \ldots, r_{k,p_k}\}$ where $r_{k,i}$ are uniformly distributed numbers with zero mean and variance $\sigma = 1/\sqrt{12}$. These random numbers are then used to create a periodic, random perturbation, with perturbation coefficients given by $C_k(m) = J r_{km'}$ where $m' = m \mod p_k$. Thus, the Hamiltonian H_k now represents random, periodic, but reflection asymmetric, long-ranged hopping between sites of the lattice that are separated by distance k.

The first two cases we consider are that of those of on-site disorder ($k = 0$) and nearest-neighbor hopping ($k = 1$).[19] The latter periodic disorder is implemented as a deviation from the constant-hopping Hamiltonian

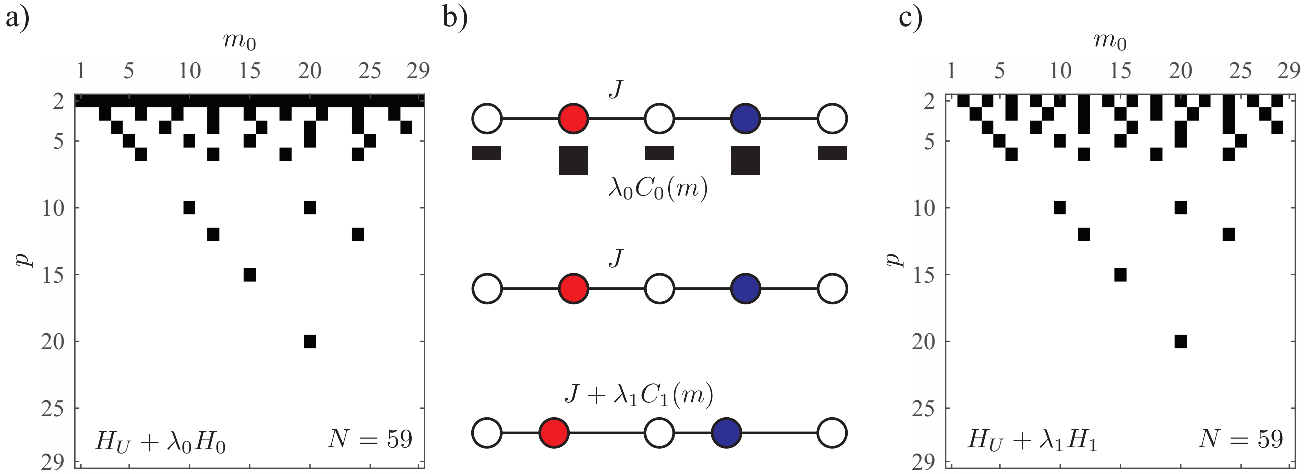

Figure 2. (a) $\mathcal{PT}$ phase diagram as a function of on-site disorder ($k = 0$) period p and gain location m_0 for an $N = 59$ site lattice. Nonzero $\mathcal{PT}$ threshold exists only at specific values (filled, black squares) of (m_0, p) that are consistent with Eqs.(6)-(7). When $p = 2$, the threshold is always positive because the underlying Hamiltonian $H_U + \lambda_0 H_0$ is $\mathcal{PT}$-symmetric. (b) Schematic of a disordered lattice. Periodic, on-site disorder $\lambda_0 C_0(m)$ with constant hopping J (top); a disorder-free lattice (center); and a lattice with $p = 2$ nearest-neighbor hopping disorder $\lambda_1 C_1(m)$ (bottom). (c) $\mathcal{PT}$ phase diagram as a function of nearest-neighbor hopping disorder ($k = 1$) period and gain location for the same $N = 59$ site lattice. Nonzero $\mathcal{PT}$ threshold exists only at specific values of (m_0, p), shown by filled, black squares.

by combining H_U with H_1 to form the Hermitian nearest-neighbor hopping matrix

$$H_{n.n.} = H_U + \lambda_1 H_1, \tag{4}$$

where the dimensionless variable λ_1 controls the strength of the disorder. Similarly, for on-site disorder, the combined Hamiltonian is given by

$$H_{o.s.} = H_U + \lambda_0 H_0, \tag{5}$$

where, again, the dimensionless variable λ_0 denotes the strength of the on-site disorder. While $H_{n.n.}$ and $H_{o.s.}$ are Hermitian, in general, they are not $\mathcal{PT}$-symmetric and therefore their real eigenfunctions are not interlaced even and odd functions. When combined with a $\mathcal{PT}$-symmetric gain-loss term Γ, the Hamiltonian $H(\gamma) = H_{n.n.} + \Gamma$ (or $H(\gamma) = H_{o.s} + \Gamma$) is neither Hermitian nor $\mathcal{PT}$ symmetric. Thus, based on perturbation-theory arguments, one might expect that its $\mathcal{PT}$-symmetry breaking threshold is zero. However, studies have found that not to be the case. Indeed, when $k = 0$ or $k = 1$,[20] it is found that the $\mathcal{PT}$-breaking threshold is positive when the lattice size N, the gain location m_0, and the disorder period p satisfy

$$N + 1 = 0 \mod p, \tag{6}$$

$$m_0 = 0 \mod p. \tag{7}$$

We note that these conditions ensure that if the gain-site index m_0 is a multiple of the disorder period p, then so is the loss-site index as well, i.e. $\bar{m}_0 = 0 \mod p$. Figure 2 summarizes the results for these two types of disorder. The center panel in Fig. 2 shows the schematic of lattices with on-site potential disorder $\lambda_0 C_0(m)$ (top), uniform lattice (center), and nearest-neighbor hopping disorder $\lambda_1 C_1(m)$ (bottom). The left-hand panel in Fig. 2 shows a two-dimensional grid in the (m_0, p) plane for an $N = 59$ site lattice, where a positive $\mathcal{PT}$ breaking threshold is denoted by a filled, black square. The right-hand panel shows corresponding results for a nearest-neighbor hopping disorder. In both cases, we see that $\gamma_{PT}(m_0, p) > 0$ only when the disorder period p and the gain location m_0 satisfy criteria in Eqs.(6)-(7).

2.3 Beam Propagation

The surprising results in Sec. 2.2 suggest that "balanced gain-loss configurations" with a positive $\mathcal{PT}$-symmetry breaking threshold are robust against certain types of periodic, random perturbations. To verify whether this

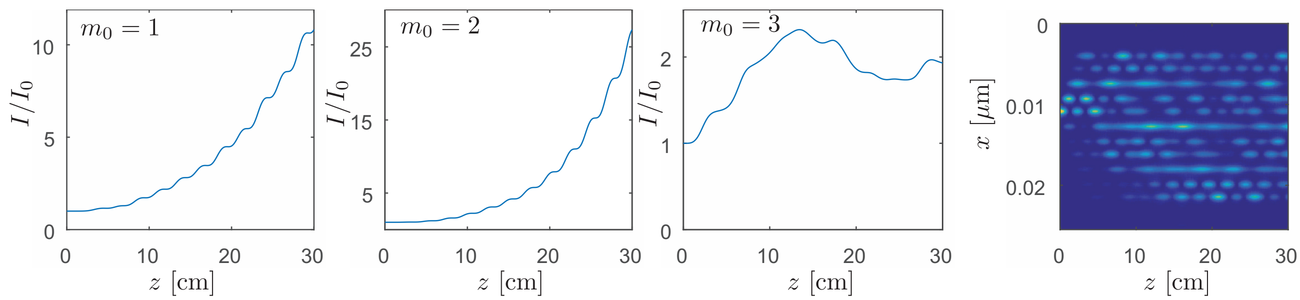

Figure 3. Beam propagation method (BPM) results for an array with $N = 11$ waveguides show the relative intensity $I(z)/I_0$ as a function of longitudinal distance along the waveguides z. The incident light (with intensity I_0) has wavelength $\lambda = 633$ nm, the cladding index of refraction is $n_0 = 1.45$, and the waveguides are 5 μm wide. Results are for a system size $N = 11$ and an on-site disorder with period $p = 3$. A balanced gain and loss with strength $\gamma = 0.5$ cm^{-1} is introduced at waveguide locations $m_0 = 1, 2, 3$ and the system is found to be in $\mathcal{PT}$-broken phase when $m_0 = 1, 2$, but in the $\mathcal{PT}$-symmetric phase for $m_0 = 3 = p$. On the far right, the local intensity pattern $I(z, x)$ is shown for $m_0 = 3$, where x is the transverse (profile) axis for the waveguide array.

property is an artifact of the tight-binding approximation or whether it is applicable to realistic experimental situations where every waveguide "site" has a transverse width over which the electric field varies, we simulate time-evolution in these systems using the beam propagation method (BPM). in these simulations, the profile of each waveguide has a finite nonzero width, and the dynamics are determined by Maxwell's equations which adequately describe the propagation of light in mixed media. For long, straight waveguides with direct light input, we may use the paraxial approximation to transform the Maxwell equations into a single, Schrödinger-like, first-order-in-time differential equation for the envelope function $\psi(x, z)$ of the electromagnetic field as a function of the transverse location x and the longitudinal distance z along the length of the waveguide,[22]

$$\mathrm{i}\frac{\partial \psi}{\partial t} = -\frac{c}{2k_0 n_0^2}\frac{\partial^2 \psi}{\partial x^2} + ck_0\left[1 - \left(\frac{n}{n_0}\right)^2\right]\psi. \tag{8}$$

Here, c is the speed of light in vacuum, k_0 is the wavenumber of incident light, n_0 is the background (cladding) index of refraction, and $n = n(x)$ is determined by the index of refraction profile. In this approximation, the distances between adjacent waveguides determine the nearest-neighbor hopping amplitude J between them and each waveguide's index of refraction $n(x)$ plays the role of on-site potential. We implement gain and loss potentials by adding imaginary parts to the index of refraction.

In a simulation with $N = 11$ waveguides based on realistic parameters,[19,23] we introduce an on-site, periodic disorder with period $p = 3$ and examine the intensity profiles $I(z, x)$ along the waveguide for gain locations $m_0 = 1, 2, 3$ (Figure 3). When $m_0 = 1$ and $m_0 = 2$, Fig. 3 shows that the spatially integrated net intensity $I(z)/I_0$ increases exponentially with time or equivalently propagation distance z, denoting that the system is in the $\mathcal{PT}$-symmetry broken phase. On the other hand, when $m_0 = 3$, we find that the net intensity $I(z)/I_0$ oscillates with time, and thus shows that the system is in the $\mathcal{PT}$-symmetric phase even in the presence of disorder that is not reflection symmetric. The last panel in Fig. 3 shows corresponding the spatio-temporal intensity profile $I(z, x)$ for an initial state localized in the fifth waveguide. Thus, we find that the tight-binding results regarding a positive $\mathcal{PT}$-symmetry breaking threshold hold up for a more realistic waveguide simulation.[19,20]

2.4 Disorder via random, periodic, long-range hopping

Until now, we have restricted ourselves to lattices with only on-site or nearest-neighbor hopping disorder. We now move to examine systems with higher degrees of connectivity, which, in the lattice model, correspond to hopping processes between sites m and $m+k$ with $k \geq 2$. Consider a lattice with both nearest-neighbor hopping determined by $\lambda_1 C_1(k)$ and next-nearest-neighbor hopping determined by $\lambda_2 C_2(k)$ as shown in Fig. 4. The Hamiltonian for the next-nearest-neighbor hopping is obtained by setting $k = 2$ in Eqn. 3. The Hermitian

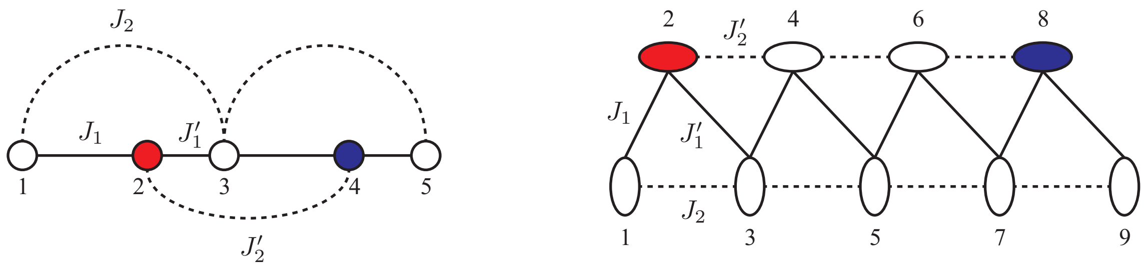

Figure 4. (Left) Depiction of a one-dimensional lattice with nearest-neighbor hopping $J + \lambda_1 C_1(m) \in \{J_1, J_1'\}$ and next-nearest-neighbor hoppings $C_2(m) \in \{J_2, J_2'\}$. This leads to a periodicity of $p_1 = 2$ and $p_2 = 2$ along the first and second off-diagonals of the pentadiagonal Hamiltonian $H = H_U + \lambda_1 H_1 + \lambda_2 H_2$ respectively. (Right) Realization of the same type of next-nearest-neighbor hopping in a two-dimensional array of elliptical waveguide with alternating vertical hopping amplitudes J_1 and J_1', and next-nearest-neighbor horizontal hopping amplitudes J_2 and J_2'. The relative orientation of the two elliptical cross-sections and the distances between the waveguides can be used to induce the requisite periodic modulations of hopping amplitudes.

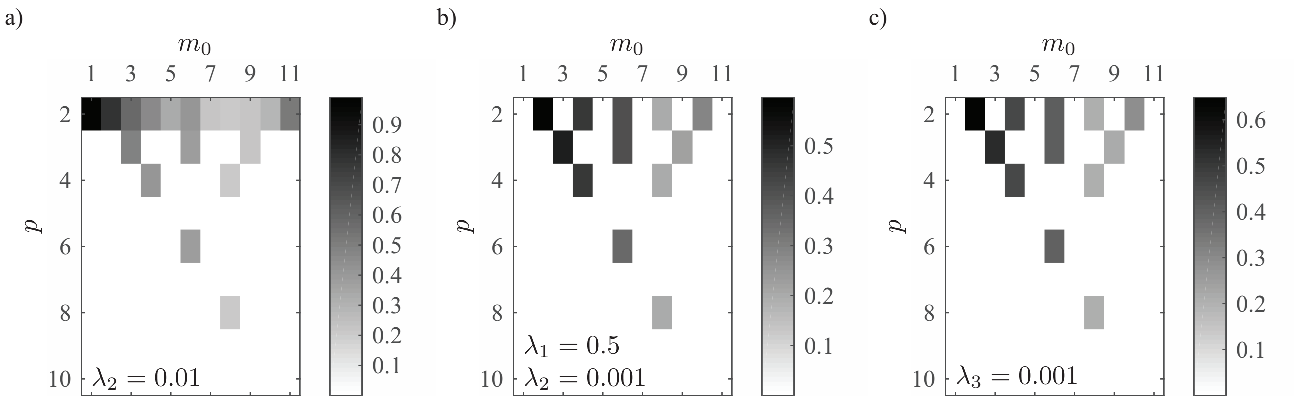

Figure 5. Phase diagram showing the $\mathcal{PT}$-breaking threshold corresponding to periodicity p and gain location m_0, results are shown averaged over 100 random realizations; the colorbar indicates the scale of the results for each panel and is measured relative to J, the uniform coupling frequency. In (a), we have uniform nearest-neighbor coupling combined with periodic disorder in next-nearest-neighbor coupling modulated by λ_2; in (b), periodic disorder is introduced in both nearest-neighbor and next-nearest-neighbor couplings (same periodicity), modulated by λ_1 and λ_2 respectively; in (c), we have uniform nearest-neighbor coupling combined with periodic disorder in third-nearest-neighbor coupling modulated by λ_3.

Hamiltonian for the disordered lattice is given by

$$H_{n.n.n.} = H_U + \lambda_1 H_1 + \lambda_2 H_2, \tag{9}$$

and the full, non-Hermitian, non-$\mathcal{PT}$-symmetric Hamiltonian is $H(\gamma) = H_{n.n.n.} + \Gamma$. Once again, since the real eigenfunctions of $H_{n.n.n.}$ are not interlaced even and odd functions, one expects the $\mathcal{PT}$-symmetry breaking threshold to be zero.

Figure 5 shows the $\mathcal{PT}$-symmetry breaking threshold γ_{PT}/J as a function of gain location m_0 and disorder period p for an $N = 23$ site lattice. The left-hand panel shows that when a uniform lattice is perturbed by random, periodic next-nearest-neighbor hopping, the $\mathcal{PT}$-threshold remains positive for (m_0, p) that satisfy Eqs.(6)-(7). The center panel shows the same pattern for a positive $\mathcal{PT}$-breaking threshold in the presence of both $k = 1$ and $k = 2$ perturbations, Eq.(3). The right-hand panel shows that even for a long-range hopping disorder, $k = 3$, an identical behavior is found. The results in Figure 5 thus show that the $\mathcal{PT}$-symmetric phase of the Hamiltonian $H(\gamma) = H_U + \Gamma$ is robust against random, periodic, long-range hopping perturbations for specific periodicities and gain locations. We note that the positive threshold values γ_{PT}/J shown in Fig. 5 are for specific realizations

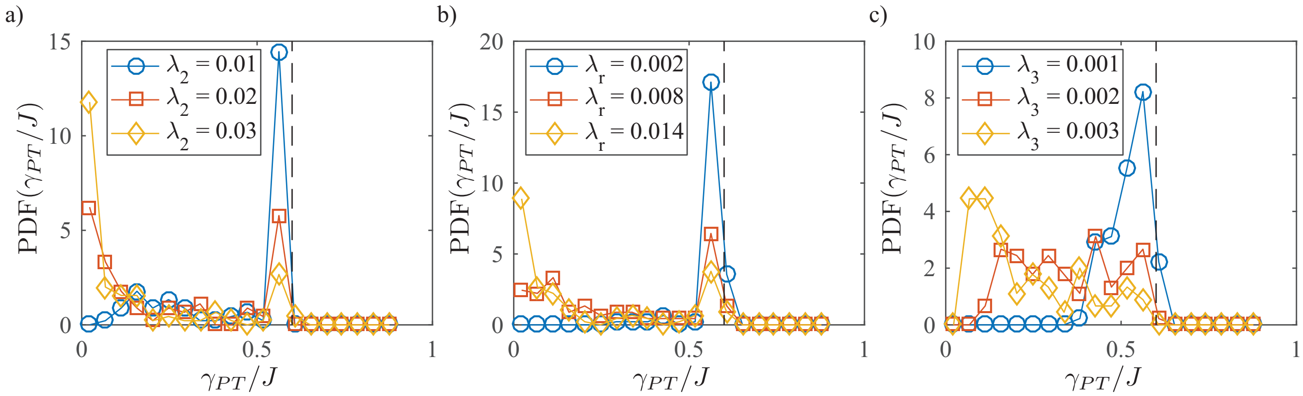

Figure 6. Probability density function (PDF) showing the distribution shift of $\gamma_{\mathcal{PT}}$ for increasingly large disorder strengths λ. In (a), we show the results for $H = H_U + \lambda_2 H_2$; in (b), we show the results for $H = H_U + \lambda_1 H_1 + \lambda_2 H_2$, with $\lambda_r = \lambda_2/\lambda_1$; and, in (c), we show the results for $H = H_U + \lambda_3 H_3$.

of the disordered Hamiltonian; thus, different disorder realizations will lead to different threshold values. What does not change, however, are the locations in the (m_0, p) plane where a positive threshold is obtained.

To quantify this variation in the threshold value for different disorder strengths λ_k, we obtain the probability distribution function (PDF) of the $\mathcal{PT}$-symmetry breaking threshold for an $N =$ site lattice. Note that the PDF depends upon the gain-site location m_0 and the distribution of the disorder.[20] The left-hand panel in Fig. 6 shows PDF(γ_{PT}/J) as a function of threshold γ_{PT}/J in the presence of only a $k = 2$ disorder, i.e. $H = H_U + \lambda_2 H_2$. We see that as λ_2 is increased, the weight in the PDF shifts towards lower values and the weight at the origin increases. The center panel shows corresponding PDF for a system with both nearest-neighbor ($k = 1$) and next-nearest-neighbor ($k = 2$) disorders. We again see the same qualitative trend. The right-hand panel shows that for a $k = 3$ disorder, the PDF broadens with increasing λ_3 and the weight shifts towards lower values but not towards the origin. These results quantify the strengths of long-range hopping disorders that leave the $\mathcal{PT}$-symmetric phase unaffected.

3. CONCLUSION

In this paper, we have investigated the effects of random, periodic, long-range hopping disorder on the fate of the $\mathcal{PT}$-symmetric phase in a uniform lattice with a pair of balanced gain-loss potentials. Although such disorder, in general, destroys the $\mathcal{PT}$-symmetry of the underlying Hermitian Hamiltonian, we find that a positive $\mathcal{PT}$-threshold is preserved when lattice size N and the gain location m_0 are both related to the disorder period p by $N+1 = 0 \mod p$ and $m_0 = 0 \mod p$. These results are valid for next-nearest-neighbor or higher-order hopping processes, and thus hint at a hidden symmetry[19,20] that preserves the positive $\mathcal{PT}$-symmetry breaking threshold of the clean lattice.

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